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Statistical aspects of the anyon model

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Abstract. The second virial coefficient for a gas of anyons is computed (i) by discretising the two-particle spectrum through the introduction of a harmonic potential regulator and (ii) by considering the problem in the continuum directly through heat kernel methods. In both cases the result of Arovas *et al* is recovered.

1. Introduction

In the last decade there has been strong interest in the study of two-dimensional systems of charged flux tubes interacting electromagnetically [1]. One of the most interesting features about these objects concerns their statistics, which has been interpreted by some authors as exotic or fractional [1, 2], interpolating between the bosonic and fermionic cases. For this reason these flux tubes have been called anyons. From a point of view of field theory, such objects can be described semiclassically since they arise as classical solutions of a Higgs model in $2+1$ dimensions with topological Chern-Simons terms [3]. The complete derivation of the statistical mechanics of a gas of anyons is an open problem that has been partially addressed by computing the second virial coefficient [4]. Furthermore, several authors have stressed the possible relevance of anyons for the fractional quantum Hall effect and two-dimensional models of high-T_c superconductors [5].

In this paper we will mainly focus on the quantum statistical mechanics of anyons. The outline of the paper is as follows: we first recall the classical picture of pairs of anyons. This is interesting for pedagogical reasons, and is also related to a recent analysis of $(2+1)$-dimensional gravity [6]. We then review the quantum mechanics of pairs of anyons possibly interacting with a harmonic potential. As for the statistics we adopt a conservative point of view where the wavefunction is single valued and the statistics of the charged flux tubes are either bosonic or fermionic according to the type of particles one has. We also analyse the case of pairs of anyons in a constant magnetic field. We then address the question of computing the second virial coefficient for a low density gas of anyons. In the original formulation of the problem [4], (i) one either puts the system in a box with appropriate boundary conditions leading to a discrete spectrum, then one takes the limit of infinite volume which gives a non-trivial finite result; or (ii) one tackles directly the path integral computation of the two-point Green function; but again a regularisation is used by inserting an appropriate convergent factor. We first show that a very simple physical interpretation can be given to the regularisation used in (ii): namely, it is equivalent to compute the two-body partition

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function of anyons interacting with a harmonic force and then to take the limit of a vanishing harmonic force. In this approach, the harmonic potential acts as a regulator that yields a discrete energy spectrum and allows us to recover the second virial coefficient in a straightforward way. It would be obviously more satisfactory to compute the second virial coefficient directly in the continuum, without relying on regularisation procedures (finite volume or harmonic well) that are in fact equivalent to discretising the spectrum. A first try consists in evaluating the partition function in terms of the scattering data of the system [7]. This approach, however, fails to give the correct two-body partition function because the interacting potential due to anyons has a long range. We then tackle directly the two-body partition function in the continuum by computing the corresponding Green function, following the work of Marino *et al* [8] based on heat kernel methods. We stress that this calculation takes place entirely in the infinite-volume limit and leads again to the well known result. This approach reveals an interesting feature, namely that the exchange terms play a very important role and indeed are not negligible at high temperature, as in the hard sphere gas case [9] (see also the Coulomb problem [10]). In our case this peculiar feature is due to the absence of a length scale (as we consider point vortices).

The model is defined in the following way: we consider identical particles of mass m carrying a charge e and a flux tube ϕ and living in a two-dimensional space. These particles are either bosons or fermions. Following [1, 2] we take for the classical two-particle Hamiltonian

$$2mH=\left(\boldsymbol{p}_1-\frac{e\phi}{\pi r}\boldsymbol{u}'\right)^2+\left(\boldsymbol{p}_2+\frac{e\phi}{\pi r}\boldsymbol{u}'\right)^2 \tag{1}$$

where $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{u}'=\boldsymbol{k}\times\boldsymbol{r}/r$, $\boldsymbol{k}$ being the unit vector orthogonal to the two-dimensional plane.

2. The classical picture

The classical picture of the system (1) was first discussed by Leinaas and Myrheim [2]. They found that the configuration space of a system of N identical particles living in an n-dimensional Euclidean space E_n is

$$E_n^N/S_N=E_n r(n,N) \tag{2}$$

where S_N is the symmetric group of permutations of N identical objects, $r(n, N)$ is the relative configuration space of nN-n dimensions, and E_n is the centre-of-mass configuration space. We recall that $r(n,N)=E_{nN-n}/S_N$. In the following we will concentrate on the case $N=2$.

In the case where $n=1$ the relative configuration space is $R^+\cup\{O\}$ with $\{O\}$ included if one allows the two particles to coincide. The interesting feature of this configuration space is that it has a boundary which allows for non-trivial interactions on it: for example, one can have a repulsive force at the origin of R^+ that acts only when the particles coincide.

In the case of interest where $n=2$, the relative configuration space is a folded cone $K=]0,\infty[\times P_1(R)\cup\{O\}$ with an apex $\{O\}$ included if one allows the two particles to coincide. Any point P on this cone corresponds to a couple of identical particles in the physical space at a distance $r=OP$ from each other. It is interesting to note that, as in the one-dimensional case, an interaction can be plugged in at the boundary of

K, which is the apex $\{O\}$. For instance, in the case of real flux tubes, one needs to remove not only the apex but also a small region around it, allowing a magnetic flux to pass through the hole. By unfolding any point on the cone we recover a good representation of a system of two anyons in a plane. Moreover, any interaction that could be introduced on the cone, like a harmonic force, will yield an interaction between the two anyons once unfolded.

The configuration space of a system of two anyons is locally equivalent to the curved (2+1)-dimensional gravity space [6]. Indeed, the scattering of a test particle of mass m in the field of a stationary point of mass m_0 gives rise to a locally flat conical metric, the angle of the cone being $\sin^{-1}\gamma$ with $1-\gamma=4Gm_0$ (G is Newton's constant). The corresponding relative classical Hamiltonian is then

$$H=\frac{m\dot{r}^2}{2\gamma^2}+\frac{L^2}{2mr^2}. \tag{3}$$

One can compare this with the relative Hamiltonian deduced from (1) which is

$$H=m\dot{r}^2+\frac{(L-e\phi/\pi)^2}{mr^2} \tag{4}$$

where L is the canonical angular momentum. Comparing the two Hamiltonians leads to identifying $\gamma^2=(L-e\phi/\pi)^2/L^2$. It is interesting to note that in this picture the angle of the cone varies with the angular momentum for a given flux. When the flux vanishes (the free case) the cone is completely open, which indeed in terms of the (2+1)-dimensional gravity leads to a flat space without matter.

3. Quantum mechanics

Let us now turn to the quantum treatment of the Hamiltonian (1). Transforming as usual to the centre of mass and relative coordinates, and leaving aside the free motion of the centre of mass, the relative Hamiltonian takes the form

$$mh_r=-\partial_r^2-\frac{1}{r}\partial_r-\frac{1}{r^2}\partial_\theta^2+2\frac{\mathrm{i}e\phi}{\pi r^2}\partial_\theta+\frac{e^2\phi^2}{\pi^2r^2}. \tag{5}$$

Parametrising the relative wavefunction as $\exp(\mathrm{i}M\theta)f(r)$ where M is standard, namely $M=2p$ for bosons and $M=2p+1$ for fermions (this assignment corresponds to a particular choice of the angular momentum operator [11]), leads to the following Schrödinger equation:

$$-mEf(r)=\left[\partial_r^2+\frac{1}{r}\partial_r-\left(M-\frac{e\phi}{\pi}\right)^2\frac{1}{r^2}\right]f(r). \tag{6}$$

Thus, as expected from the classical analysis, (6) is nothing but the relative Hamiltonian of a free particle where the angular momentum M is replaced by $M'=M-\alpha$ with $\alpha=e\phi/\pi$. It follows immediately that the spectrum of h_r is continuous and that the radial wavefunction is $J_{|M'|}(kr)$. (Note again the analogy with the (2+1)-dimensional gravity case where the wavefunction is $J_{|M|/\gamma}(kr)$ [6].)

In the case where a harmonic potential term $m(\omega r/2)^2$ is present a standard calculation yields the following energy levels [1, 2] (energy, degeneracy): $((2j+1+2\Delta)\omega, j+1)$ and $((2j+1-2\Delta)\omega, j)$ where j is a positive or null integer. Δ stands for the fractional part of $e\phi/2\pi$ or $e\phi/2\pi+\frac{1}{2}$, depending on whether the particles are bosons or fermions. The usual bosonic case without flux $((2j+1)\omega, 2j+1)$ is recovered by letting $\Delta=0$ and the fermionic case $((2j)\omega, 2j)$ by letting $\Delta=\frac{1}{2}$.

It is also interesting to investigate the case where one adds an external orthogonal magnetic field to the system. The centre-of-mass Hamiltonian describes the motion of a $(2m, 2e)$ particle orbiting in the orthogonal magnetic field. In the symmetric gauge where the vector potential takes the form $\boldsymbol{A}=-\boldsymbol{r}\times\boldsymbol{B}/2$ the Schrödinger equation for the relative motion becomes

$$-mEf(r)=\left(\partial_r^2+\frac{1}{r}\partial_r-\frac{M'^2}{r^2}+\frac{eB}{2}M'-\left(\frac{e}{2}\right)^2\frac{B^2r^2}{4}+\frac{m^2\omega^2r^2}{4}\right)f(r). \qquad (7)$$

As above, the flux tubes shift the orbital angular momentum from M to M'. It is clear that the external magnetic field alters the spectrum dramatically. For instance in the case where ω is set to zero (a particular case of interest where the harmonic well term is entirely due to the magnetic field) one gets for the relative motion $((j+\frac{1}{2})eB/m, \infty)$ and $((j+\frac{1}{2}+2\Delta)eB/m, 1+j/2$ for j even and $1+(j-1)/2$ for j odd). In the case where the ω-regulator is present the degeneracy is completely lifted and one obtains the energy levels $[n+\frac{1}{2}+(|p|\pm\Delta)(1\pm B/B')]eB'/m$ with $n, |p|$ integers $\geqslant 0$ and where the $\pm$ signs refer to $p\geqslant 0$ or $p<0$ accordingly $(B'^2-B^2\equiv(2m\omega/e)^2)$.

4. Statistical mechanics: the second virial coefficient

Let us first recall the basic definition of the second virial coefficient [12]. For $\beta=1/kT$ the grand partition function is defined in terms of the fugacity z by

$$\Xi=\sum_{N=0}^{\infty}\frac{z^N}{N!}\,\mathrm{Tr}\exp(-\beta H_N) \qquad (8a)$$

where H_N denotes the Hamiltonian for a N-particle system. In a low density approximation, we can use the cluster expansion:

$$\Xi=\exp\left(V\sum_{l=1}^{\infty}b_l z^l\right). \qquad (8b)$$

Here V is the volume and b_l stands for the lth cluster integral. Comparing (8a) and (8b) leads to

$$b_1=\frac{1}{V}\,\mathrm{Tr}\exp(-\beta H_1) \qquad (9a)$$

$$b_2=\frac{1}{2V}\{\mathrm{Tr}\exp(-\beta H_2)-[\mathrm{Tr}\exp(-\beta H_1)]^2\}. \qquad (9b)$$

The equation of state for a real gas expanded in powers of the density ρ is

$$PV=\frac{N}{\beta}(1+a_2\rho+a_3\rho^2+\ldots) \qquad (10)$$

where the a_i stand for the virial coefficients. The computation of $a_2(T)$ alone requires the knowledge of two-body interactions. Indeed one has

$$a_2=-b_2/b_1^2. \qquad (11)$$

In two spatial dimensions $b_1 = 1/\lambda^2$ where λ is the thermal wavelength of a particle of mass m and is given by $\lambda = (2\pi\beta/m)^{1/2}$. Let us introduce the two-point Green function for the relative motion $G(\boldsymbol{r}, \boldsymbol{r}') = \langle \boldsymbol{r}'|\exp(-\beta h_{\text{rel}})|\boldsymbol{r}\rangle$. One has $b_2 = (1/\lambda^2)\int d^2\boldsymbol{r}\, G(\boldsymbol{r}, \boldsymbol{r}) - V/2\lambda^4$ (in this formula the centre-of-mass motion has been extracted). It follows that $a_2 = -\lambda^2 \int d^2\boldsymbol{r}\, G(\boldsymbol{r}, \boldsymbol{r}) + V/2$ where G has to be properly symmetrised or antisymmetrised. For example, in the free case one obtains

$$a_2 = -\lambda^2 \int d^2\boldsymbol{r}(G_0(\boldsymbol{r}, \boldsymbol{r}) \pm G_0(\boldsymbol{r}, -\boldsymbol{r})) + V/2 = \mp \frac{\pi\beta}{2m} \tag{12a}$$

with

$$G_0(\boldsymbol{r}, \boldsymbol{r}') = \frac{m}{4\pi\beta} \exp[-m(\boldsymbol{r} - \boldsymbol{r}')^2/4\beta] \tag{12b}$$

where $\mp$ signs refer to bosons or fermions respectively. If one lets $G_{\text{int}} = G - G_0$, the second virial then takes the form:

$$a_2 = \mp \frac{\pi\beta}{2m}\left(1 \pm 4 \int d^2\boldsymbol{r}(G_{\text{int}}(\boldsymbol{r}, \boldsymbol{r}) \pm G_{\text{int}}(\boldsymbol{r}, -\boldsymbol{r}))\right). \tag{13}$$

In the previous section we have seen that, if we introduce above the long-range potential due to the flux tubes, an ω-harmonic potential betwen the anyons, then the spectrum becomes entirely discrete. The purpose of what follows is to show that in the limit ω goes to 0 (which ensures the transition to a continuous spectrum) one can obtain a finite result for the second virial coefficient. Indeed the regularised relative two-body partition function is in this case

$$\begin{aligned} Z_\omega(\Delta) &= \sum_{j=0}^{\infty} \{(j+1)\exp[-\beta\omega(2j+1+2\Delta)] + j\exp[-\beta\omega(2j+1-2\Delta)]\} \\ &= \frac{1}{2}\frac{\cosh \beta\omega(2\Delta-1)}{\sinh^2 \beta\omega}. \end{aligned} \tag{14}$$

Z_{int} must vanish when the electromagnetic interaction between the anyons is switched off: it follows that it must be defined as

$$Z_{\text{int}} = \tfrac{1}{2} \int d^2\boldsymbol{r}(G_{\text{int}}(\boldsymbol{r}, \boldsymbol{r}) + G_{\text{int}}(\boldsymbol{r}, -\boldsymbol{r})) = \lim_{\omega \to 0}(Z_\omega(\Delta) - Z_\omega(0)) \tag{15a}$$

in the case of bosons and

$$Z_{\text{int}} = \tfrac{1}{2} \int d^2\boldsymbol{r}(G_{\text{int}}(\boldsymbol{r}, \boldsymbol{r}) - G_{\text{int}}(\boldsymbol{r}, -\boldsymbol{r})) = \lim_{\omega \to 0}(Z_\omega(\Delta) - Z_\omega(\tfrac{1}{2})) \tag{15b}$$

in the case of fermions. From (13) and (14) one thus obtains for the second virial coefficient respectively:

$$a_2 = -\frac{\pi\beta}{2m}[1 + 8\Delta(\Delta - 1)] \tag{16a}$$

$$a_2 = \frac{\pi\beta}{2m}[1 - 8(\Delta - \tfrac{1}{2})^2]. \tag{16b}$$

These results coincide with the expressions derived in [4]: moreover, the harmonic well regularisation is in fact equivalent to the 'path integral' approach used in [4].

Indeed it is not difficult to realise that (14) follows from equations (22) and (23) [4] and by identifying $a = 1 + \varepsilon = \cosh \beta\omega$.

Naturally, it would be more satisfactory to obtain this result directly in the continuum. One way out could be to follow Uhlenbeck and Beth [7], and to express $Z_{\rm int}$ in terms of the scattering data for the corresponding relative two-body problem. In two spatial dimensions, denoting by $\delta_m(k)$ the phase shift of the partial wave of momentum k and angular momentum m, one has (m is even for bosons, odd for fermions)

$$Z_{\rm int} = \frac{1}{\pi} \int_0^\infty \exp(-\beta k^2) \sum_m \frac{{\rm d}\delta_m}{{\rm d}k} \, {\rm d}k. \tag{17}$$

It is clear that since the phase shifts $|\delta_m(k)| = \pi\alpha/2$ are momentum independent, (17) seems to give the naive result $Z_{\rm int} = 0$. The Uhlenbeck–Beth formula leads to this paradoxical result because of the long-range character of the two-body potential, a regime for which a treatment in terms of scattering data is known to be singular [13] (actually we would encounter the same difficulty in three dimensions with the same type of potential). The origin of this paradox can be traced back to the fact that $Z_{\rm int}$ does not depend on the temperature: this means that the introduction of a long-range potential amounts to no more than altering the low energy part of the density of states. Indeed, the shift in the density of states $\rho(k) - \rho_0(k) = (1/\pi) \sum_m {\rm d}\delta_m/{\rm d}k$, which is nothing more than the inverse Laplace transform of the two-body partition function, is equal to $\Delta(\Delta - 1)\delta(k)$ and is thus concentrated at the bottom of the spectrum. Naturally, this result cannot be reached by a naive counting of modes in the continuum, which is the basis of the derivation of (17).

Let us now compute the two-point Green function $G(\boldsymbol{r}, \boldsymbol{r}', \beta)$ directly in the continuum. Its Laplace transform $G(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) = \int_0^\infty {\rm d}\beta \, G(\boldsymbol{r}, \boldsymbol{r}', \beta) \exp[-(\beta\mathcal{M}^2)]$ is given by

$$G(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) = \frac{1}{2\pi} \int_0^\infty \frac{k \, {\rm d}k}{k^2 + \mathcal{M}^2} \left(\sum_{m=0}^{\infty} \exp({\rm i}m\Delta\theta) J_{m+\alpha}(kr) J_{m+\alpha}(kr') + \sum_{m=1}^{\infty} \exp(-{\rm i}m\Delta\theta) J_{m-\alpha}(kr) J_{m-\alpha}(kr') \right) \tag{18}$$

where $\Delta\theta = \theta - \theta'$. Without loss of generality it has been assumed that $0 \leqslant \alpha < 1$. Equation (18) is obviously periodic in α with a period equal to 1. Following Marino *et al* [8], it is tedious but straightforward to rewrite (18) as

$$G(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) = -\frac{\sin \pi\alpha}{(2\pi)^2} \int_{R^2} {\rm d}v \, {\rm d}w \frac{\exp[\alpha(v - w)]}{1 + \exp({\rm i}\Delta\theta + v - w)} \times \exp(-\mathcal{M}r \cosh w - \mathcal{M}r' \cosh v) + G^{\alpha}_{\rm pole}(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) \tag{19}$$

where $G^{\alpha}_{\rm pole}(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M})$ is given by

$$\frac{1}{2{\rm i}(2\pi)^2} \lim_{a \to \infty} \int_{-\infty}^{\infty} {\rm d}v \int_C {\rm d}w \frac{\exp[\alpha(v - w)]}{1 - \exp({\rm i}\Delta\theta + v - w)} \exp(\mathcal{M}r \cosh w - \mathcal{M}r' \cosh v). \tag{20}$$

Here the contour integration C in the complex ω-plane is a rectangle $[-a, a]$ along the real axis and $[-{\rm i}\pi, {\rm i}\pi]$ along the imaginary axis. We stress that $G^{\alpha}_{\rm pole}$ depends on

α as soon as $\boldsymbol{r} \neq \boldsymbol{r}'$, which is a significant difference with [8] where G^{α}_{pole} is actually G_0. Indeed, in our case one gets

$$\begin{aligned} G^{\alpha}_{\text{pole}}(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) &= \exp(\mathrm{i}\alpha\Delta\theta) G_0(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) \qquad &\text{for } \Delta\theta < \pi \\ G^{\alpha}_{\text{pole}}(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) &= \cos \alpha\pi G_0(\boldsymbol{r}, \boldsymbol{r}', \mathcal{M}) \qquad &\text{for } \Delta\theta = \pi. \end{aligned} \tag{21}$$

We are mainly interested in evaluating (18) in the cases $\boldsymbol{r} = \pm \boldsymbol{r}'$ which are the only relevant ones for the computation of the second virial coefficient according to (13). From (19) it thus follows that

$$\begin{aligned} G_{\text{int}}(\boldsymbol{r}, \pm\boldsymbol{r}, \mathcal{M}) = -\frac{\sin \pi\alpha}{(2\pi)^2} \int_{R^2} \mathrm{d}v\, \mathrm{d}w \frac{\exp[\alpha(v-w)]}{1+\exp(\mathrm{i}\Delta\theta + v - w)} \\ \times \exp(-\mathcal{M}r(\cosh w + \cosh v)] + G_0(\boldsymbol{r}, \pm\boldsymbol{r}, \mathcal{M})(\cos \alpha\Delta\theta - 1) \end{aligned} \tag{22}$$

where $\Delta\theta = 0(\pi)$ when $\boldsymbol{r} = \boldsymbol{r}'$ $(\boldsymbol{r} = -\boldsymbol{r}')$. Note that for anticoinciding points the factor $\cos \alpha\pi - 1$ is in complete agreement with the corresponding singularity in the forward scattering amplitude obtained by Ruijsenaars [13].

In (22) the free part has been explicitly subtracted from the two-body interacting part in the two-point Green function, as it should be, but we still have to evaluate the remaining integrals. After some simple manipulations and a trivial integration on $\boldsymbol{r}$, these contributions can be recast as

$$-\frac{\sin \pi\alpha}{2\pi} \frac{1}{\mathcal{M}^2} \int_{-\infty}^{\infty} \mathrm{d}\delta \frac{\exp(-2\alpha\delta)}{1 \pm \exp(-2\delta)} \frac{1}{\cosh^2 \delta} \tag{23}$$

where the $\pm$ signs refer to the cases $\boldsymbol{r} = \pm\boldsymbol{r}'$ $(s \equiv (w+v)/2$ and $\delta \equiv (w-v)/2)$. Both contributions can be computed in the same way through integration by parts, with the notable difference that in the case $\boldsymbol{r} = -\boldsymbol{r}'$ the integral is divergent at the origin. This means that some regularisation procedure has to be used to render this integral finite. According to (13) the $\int \mathrm{d}^2\boldsymbol{r}(G_{\text{int}}(\boldsymbol{r}, \boldsymbol{r}) \pm G_{\text{int}}(\boldsymbol{r}, -\boldsymbol{r}))$ contributions to a_2 that one has to compute can be written as

$$-\frac{\sin \pi\alpha}{2\pi} \frac{1}{\mathcal{M}^2} \int_{-\infty}^{\infty} \mathrm{d}\delta \frac{\exp(a\delta)}{\sinh 2\delta} \frac{1}{\cosh^2 \delta} \tag{24}$$

where $a = 2 - 2\alpha$ $(a = 2\alpha)$ in the bosonic case (fermionic case). Choosing the principal-value prescription one obtains for the bosonic and fermionic cases respectively:

$$\frac{1}{2\mathcal{M}^2}\left((1-\alpha)^2 - \cos^2 \frac{\pi\alpha}{2}\right) \tag{25a}$$

$$\frac{1}{2\mathcal{M}^2}\left(\alpha^2 - \sin^2 \frac{\pi\alpha}{2}\right). \tag{25b}$$

This last result, considered together with the contribution of the pole integral in (22), leads through an inverse Laplace transform to the following final expressions for the second virial coefficient:

$$-\frac{\pi\beta}{2m}\{1 + 2[(1-\alpha)^2 - 1]\} \tag{26a}$$

$$\frac{\pi\beta}{2m}(1 - 2\alpha^2). \tag{26b}$$

These results coincide with those derived in the discrete spectrum case (16*a*, *b*) through the identifications $\alpha = 2\Delta$ in the bosonic case and $\alpha = 2\Delta - 1$ in the fermionic case (note that (26*b*) can be obtained from (26*a*) by replacing α by $1+\alpha$ and that the allowed ranges of α are now $0 \leqslant \alpha < 2$ and $-1 \leqslant \alpha < 1$, respectively, corresponding to summations over even or odd m in (18)). We stress again the essential role played by the $\boldsymbol{r} = -\boldsymbol{r}'$ contributions, usually exponentially negligible at high temperature. As already mentioned above, this is due to the fact that there is no length scale in the model. Had we assumed that the flux tubes are impenetrable disks of radius R, the integration on space in (22) would have produced an exponential damping in (23) for the exchange term.

5. Conclusion

We have calculated the second virial coefficient for a gas of anyons directly in the continuum, reproducing the result of Arovas *et al* [4]. Out method, based on the heat kernel methods developed in [8], clearly enlightens the role of the exchange terms, due to the long-range potential. We have also shown how the regularisation used in the original path integral calculation of [4] is equivalent to adding a harmonic interaction between the anyons, leading to a discrete spectrum. In the continuum another possible approach is by using scattering data to compute the two-body partition function. However, we have seen that, if one relies on the usual naive prescription, the partition function seems to vanish trivially. It would certainly be gratifying to extend the formula of Uhlenbeck and Beth [7] to the case of long-range potentials and to get a deeper understanding of the peculiar features of the two-body density of states.

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